International Journal for Innovative Engineering and Management Research
PEER REVIEWED OPEN ACCESS INTERNATIONAL JOURNAL
www.ijiemr.org

Implementation of HVIT for Parabolic-Hyperbolic Differential Equation

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Abstract

In this Paper, He's Variational iteration technique (HVIT) is applied to evaluate nonlinear Parabolic-Hyperbolic (NLPH) Differential Equations. The HVIT gives solution closed to exact solution of nonlinear partial differential (NPD) equations, if choice of initial approximation is better. This technique is easily applied to the NLPH Differential Equations. Numerical examples are given to demonstrate solution of NLPH Differential Equations. The analytical results are in terms of convergent series which is easily computable. Numerical result shows that HVIT gives high accuracy. Also, this technique is reliable and efficient for solving wide class of NLPH Differential Equations.

Keywords: Parabolic-Hyperbolic Differential Equation, He's Variational iteration technique, Lagrange multiplier, correction functional, nonlinear Partial differential equation.

1. Introduction:

The variational iteration technique [1,3,4] was first proposed by Ji-Huan He in 1999.The VIT does not involve discretization also it does not require large number of computer memory. Therefore, it is preferable to many numerical problems as it is free from rounding off errors. The VIT is reliable and efficient for wide variety of scientific and engineering applications. This method is more powerful than Adomian decomposition method (ADM), Homotopy Perturbation method (HPM) [2] .ADM involves computations of Adomian polynomials while perturbation methods suffer from computational workload when degree of nonlinearity increases. This method is applied to linear, nonlinear, ordinary and partial differential equations [6]. This method solves large class of nonlinear problems [5] easily, effectively and accurately with approximations converging rapidly to exact solutions.

In this paper, HVIT is apply for solving some NLPH Differential Equations. The Paper has been organized as follows: In section 2, HVIT is introduced. Section 3 deals with numerical solution of NPHD equations by using HVIT. The last section deals with conclusion.

2. He's Variational Iteration Technique (HVIT):

The variational iteration technique works on nonlinear problem as similar to use for linear. There is no need to find Adomian polynomials. The main part is to determine the Lagrange multiplier. Then successive approximation can be obtained in a recursive manner. The main steps of the method are as follows:

Consider the differential equation

$$Av + Bv = C(x,t), \tag{1}$$

where A and B are linear and nonlinear operators respectively, and $C(x,t)$ is the source inhomogeneous term.

The variational iteration technique admits the correction functional for Equation (1) in the form,

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s)\big(Av_n(s) + B\bar{v}_n(s) - C(s)\big)ds, \qquad n \geq 0, \tag{2}$$

where λ is general Lagrange multiplier [3,4], The subscript n denotes nth approximation and $\bar{v}_n$ is a restricted variation which means $\delta\bar{v}_n = 0$.

The successive approximations $v_{n+1,}n \geq 0$, of the solution $v = \lim_{n\to\infty} v_n$ will be obtained using any selective function v_0 by determining the Lagrange Multiplier $\lambda(s)$.

3. The Nonlinear Parabolic-Hyperbolic Differential Equation:

A nonlinear PH differential equation [7, 8] is given as,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\partial^2}{\partial t^2} - \Delta\right)v = G(v)$$

With initial condition,

$$\frac{\partial^i v(X,0)}{\partial t^i} = g_i(X)\ ;\ X = (x_1, x_2, \dots, x_n)\ ;\ i = 0,1,2,\dots$$

Where $\Delta = \frac{\partial^2}{\partial x^2}$ = Laplacian operator and $G(v)$ is nonlinear term.

For implementation of HVIT for NLPH differential equation we will discuss following numerical examples.

Numerical Results:

Illustration-I:

Consider NLPH Differential equation in the form,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{\partial^2}{\partial t^2}-\Delta\right)v \;=\; \left(\frac{\partial^2 v}{\partial t^2}\right)^2+\left(\frac{\partial^2 v}{\partial x^2}\right)^2-2v^2\ ;\ \ 0\le x,t\le 1$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)v \;=\; \left(\frac{\partial^2 v}{\partial t^2}\right)^2+\left(\frac{\partial^2 v}{\partial x^2}\right)^2-2v^2$$

with initial condition $v(x,0)=v_t(x,0)=v_{tt}(x,0)=e^x$

We can rewrite this equation as,

$$\frac{\partial^3 v}{\partial t^3}-\frac{\partial^3 v}{\partial t\partial x^2}-\frac{\partial^4 v}{\partial x^2\partial t^2}+\frac{\partial^4 v}{\partial x^4} \;=\; \left(\frac{\partial^2 v}{\partial t^2}\right)^2+\left(\frac{\partial^2 v}{\partial x^2}\right)^2-2v^2$$

$$v_{ttt}-v_{txx}-v_{xxtt}+v_{xxxx}=(v_{tt})^2+(v_{xx})^2-2v^2$$

Now we can construct correction functional for this differential equation as,

$$v_{n+1}=v_n+\int_0^t \lambda\begin{bmatrix}(v_n)_{sss}-(\overline{v_n})_{sxx}-(\overline{v_n})_{xxss}+(\overline{v_n})_{xxxx}\\ -((v_n)_{ss})^2-((\overline{v_n})_{xx})^2+2\overline{v_n}^2\end{bmatrix}ds \tag{3}$$

Where $\overline{v_n}$ is restricted variation. therefore $\delta\overline{v_n}=0$.Also $\lambda(s)$ is Lagrange multiplier which can be obtain by using variational theory as $\lambda(s)=-\frac{1}{2}(s-t)^2$.

Therefore, correction functional (3) becomes,

$$v_{n+1}=v_n-\frac{1}{2}\int_0^t (s-t)^2\begin{bmatrix}(v_n)_{sss}-(\overline{v_n})_{sxx}-(\overline{v_n})_{xxss}+(\overline{v_n})_{xxxx}\\ -((v_n)_{ss})^2-((\overline{v_n})_{xx})^2+2\overline{v_n}^2\end{bmatrix}ds \tag{4}$$

By Taylor series expansion, $v_0(x,t)=v(x,0)+tv'(x,0)+\frac{t^2}{2!}v''(x,0)+\cdots$

By using initial condition, we get

$$v_0(x,t)=e^x+te^x+\frac{t^2}{2!}e^x$$

$$=\left(1+t+\frac{t^2}{2!}\right)e^x$$

Hence, from equation (3.94) we get

$$v_1(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)e^x$$

$$v_2(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right)e^x$$

Continuing in this way we get, $v_n = \left(\sum_{k=0}^{n+2} \frac{t^k}{k!}\right) e^x$

Hence exact solution of nonlinear PH differential equation is

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\sum_{k=0}^{n+2} \frac{t^k}{k!}\right) e^x = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right) e^x = e^t e^x = e^{x+t}$$

Illustration-II:

Two dimensional nonlinear Parabolic-Hyperbolic (NLPH) differential equation is,

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2}\right) v = \frac{\partial v}{\partial t} - 2v \tag{5}$$

With initial condition

$$v(x_1, x_2, 0) = \sinh(x_1 + x_2),$$

$$\frac{\partial v(x_1, x_2, 0)}{\partial t} = 2\sinh(x_1 + x_2),$$

$$\frac{\partial^2 v(x_1, x_2, 0)}{\partial t^2} = 4\sinh(x_1 + x_2),$$

We can write equation (5) as,

$$\frac{\partial^3 v}{\partial t^3} - \frac{\partial^3 v}{\partial t \partial {x_1}^2} - \frac{\partial^3 v}{\partial t \partial {x_2}^2} - \frac{\partial^4 v}{\partial {x_1}^2 \partial t^2} + \frac{\partial^4 v}{\partial {x_1}^4} + \frac{\partial^4 v}{\partial {x_1}^2 \partial {x_2}^2} - \frac{\partial^4 v}{\partial {x_2}^2 \partial t^2} + \frac{\partial^4 v}{\partial {x_2}^2 \partial {x_1}^2} + \frac{\partial^4 v}{\partial {x_2}^4}$$
$$= \frac{\partial v}{\partial t} - 2v$$

Now we can construct correction functional for differential equation (3.95) by using Lagrange multiplier $\lambda(s) = -\frac{1}{2}(s-t)^2$ as previous we get,

$$v_{n+1} = v_n - \frac{1}{2}\int_0^t (s-t)^2 \begin{bmatrix} (v_n)_{sss} - (\overline{v_n})_{sx_1x_1} - (\overline{v_n})_{sx_2x_2} \\ -(\overline{v_n})_{x_1x_1ss} + (\overline{v_n})_{x_1x_1x_1x_1} + (\overline{v_n})_{x_1x_1x_2x_2} \\ -(\overline{v_n})_{x_2x_2ss} + (\overline{v_n})_{x_2x_2x_1x_1} + (\overline{v_n})_{x_2x_2x_2x_2} \\ -(\overline{v_n})_s + 2\overline{v_n} \end{bmatrix} ds \tag{6}$$

By using Taylor series expansion,

$$v_0(x_1,x_2,t) = v(x_1,x_2,0) + tv'(x_1,x_2,0) + \frac{t^2}{2!}v''(x_1,x_2,0) + \cdots$$

$$= \sinh(x_1+x_2) + 2t\sinh(x_1+x_2) + 4\frac{t^2}{2!}\sinh(x_1+x_2)$$

$$= \left(1 + 2t + \frac{(2t)^2)}{2!}\right)\sinh(x_1+x_2)$$

Values of $v_1,\ v_2, \dots, v_n$ can be calculated by using equation (6) as

$$v_1(x_1,x_2,t)\ = \left(1 + 2t + \frac{(2t)^2)}{2!} + \frac{(2t)^3)}{3!}\right)\sinh(x_1+x_2)$$

$$v_2(x_1,x_2,t)\ = \left(1 + 2t + \frac{(2t)^2)}{2!} + \frac{(2t)^3)}{3!} + \frac{(2t)^4)}{4!}\right)\sinh(x_1+x_2)$$

Continuing in this way we get, $v_n(x_1,x_2,t) = \left(\sum_{k=0}^{n+2}\frac{(2t)^k}{k!}\right)\sinh(x_1+x_2)$

Hence exact solution of NLPH differential equation is

$$v = \lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n+2}\frac{(2t)^k}{k!}\right)\sinh(x_1+x_2)$$

$$= \left(\sum_{k=0}^{\infty}\frac{(2t)^k}{k!}\right)\sinh(x_1+x_2)$$

$$= \left(1 + 2t + \frac{(2t)^2)}{2!} + \frac{(2t)^3)}{3!} + \cdots\right)\sinh(x_1+x_2)$$

$$v(x,t) = e^{2t}\sinh(x_1+x_2)$$

which is exact solution of NLPH equation (3.95).

Illustration-III:

Consider two dimensional NLPH differential equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\partial^2}{\partial t^2} - \Delta\right)v = v\frac{\partial^2 v}{\partial t^2} + 3v \qquad (7)$$

with initial condition $v(x,0) = cos\,x\,, \frac{\partial v(x,0)}{\partial t} = 3\,cos\,x\,, \frac{\partial^2 v(x,0)}{\partial t^2} = 9\,cos\,x$

Equation (7) can be also written as

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)v = v\frac{\partial^2 v}{\partial t^2} + 3v$$

$$\frac{\partial^3 v}{\partial t^3} - \frac{\partial^3 v}{\partial t \partial x^2} - \frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\partial^4 v}{\partial x^4} = v\frac{\partial^2 v}{\partial t^2} + 3v$$

$$v_{ttt} - v_{txx} - v_{xxtt} + v_{xxxx} = vv_{tt} + 3v$$

Construct correction functional for this differential equation with Lagrange multiplier $\lambda(s) = -\frac{1}{2}(s-t)^2$ we get,

$$v_{n+1} = v_n - \frac{1}{2}\int_0^t (s-t)^2 \begin{bmatrix} (v_n)_{sss} - (\overline{v_n})_{sxx} - (\overline{v_n})_{xxss} \\ +(\overline{v_n})_{xxxx} - \overline{v_n}(\overline{v_n})_{ss} - 3\overline{v_n} \end{bmatrix} ds \qquad (8)$$

By using initial condition and Taylor series expansion we get

$$v_0(x,t) = v + tv' + \frac{t^2}{2!}v''$$

$$= \cos x + 3t\cos x + \frac{(3t)^2}{2!}\cos x$$

$$= \left(1 + 3t + \frac{(3t)^2}{2!}\right)\cos x$$

By using equation (8) we get

$$v_1(x,t) = v_0(x,t) - \frac{1}{2}\int_0^t (s-t)^2 \begin{bmatrix} (v_0)_{sss} - (\overline{v_0})_{sxx} - (\overline{v_0})_{xxss} \\ +(\overline{v_0})_{xxxx} - \overline{v_0}(\overline{v_0})_{ss} - 3\overline{v_0} \end{bmatrix} ds$$

$$= v_0(x,t) - \frac{1}{2}\int_0^t (s-t)^2\,(-27\,cos\,x)ds$$

$$= v_0(x,t) + \frac{27}{2}cos\,x\left[\frac{(s-t)^3}{3}\right]_{s=0}^{s=t}$$

$$= \left(1 + 3t + \frac{(3t)^2}{2!}\right)cos\,x + \frac{27}{6}t^3 cos\,x$$

$$= \left(1 + 3t + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!}\right)cos\,x$$

Now, $v_2(x,t) = \left(1 + 3t + \frac{(3t)^2}{2!} + \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!}\right)cos\,x$

Hence, we get, $v_n(x,t) = \left(\sum_{k=0}^{n+2}\frac{(3t)^k}{k!}\right)cos\,x$

Hence exact solution of nonlinear PH differential equation (7) is

$$v(x,t)=\lim_{n\to\infty} v_n=\left(\sum_{k=0}^{\infty}\frac{(3t)^k}{k!}\right)\cos x\ =e^{3t}\cos x$$

4. Conclusions:

In this article, we have used HVIT for solving initial value problem which is nonlinear Parabolic-Hyperbolic Differential Equations. The result shows that selection of initial approximation can be easily done with unknown constants. The result obtained by HVIT has good agreement with exact solution. Therefore, this technique is more powerful for obtaining accurate result. Requirement of discretization and interpolation are not necessary in VIT. Hence, it is concluded that VIT is most powerful, reliable and efficient for finding solution of NLPH differential equations.

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