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Edge Cordial Labeling of Subdivision Graphs of Some Standard Graphs

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Abstract

In this paper, we investigate the edge cordiality of subdivision graphs obtained from several standard graph families. For a graph G, its subdivision graph $S(G)$is obtained by inserting a new vertex into every edge of G. We study the edge cordial labeling of $S(G)$ for paths, cycles, wheel graph, complete graph, comb graph and sunlet graph. Necessary labeling conditions are established and explicit balanced edge-labeling constructions are described. In particular, it is shown that the subdivision graph of a path is edge cordial for every n, whereas the subdivision graph of a cycle C_n is edge cordial precisely when n is even. For complete graphs, the small cases K_3 and K_4 provide exceptions, while $S(K_3)$ $S(K_N)$is edge cordial for n. The subdivision graphs of comb graphs and sunlet graphs admit edge cordial labelings for their natural parameter ranges. The results demonstrate how subdivision changes the parity structure governing cordiality.

Keywords: Edge cordial labeling, cordial graph, subdivision graph, path, cycle, wheel, complete graph, comb graph, sunlet graph.

1. Introduction

Graph labeling is an important area of graph theory in which the vertices or edges of a graph are assigned labels subject to specified arithmetic or combinatorial conditions. Among the various types of graph labelings, cordial labeling is particularly useful because it imposes a balance condition on the distribution of labels.

An edge cordial labeling is a binary labeling of the edges of a graph such that the numbers of edges receiving the two labels differ by at most one, while the induced vertex labels are also nearly equally distributed. The concept becomes particularly interesting when the labeling is considered on graph transformations. One of the fundamental graph transformations is the **subdivision operation**.

Given a graph G, its subdivision graph $S(G)$ is obtained by replacing every edge uv by a path $u - x_{uv} - v$ of length two, where x_{uv} is a newly introduced vertex.Subdivision changes both the order and size of a graph and consequently changes the parity conditions involved in cordial labeling. In this paper, we examine this effect for several important graph families: P_n, C_n, W_n, K_n as well as comb graphs and sunlet graphs.

The principal objective is to determine when their subdivision graphs admit an edge cordial labeling

2. Preliminaries

Let $G = (V, E)$ be a finite simple graph with $|V(G)| = p, |E(G)| = q$

Definition 2.1 : For graph G, the edge labelling function is defined as $f: E(G) \to \{0,1\}$ and induced vertex labeling function $f^*: V(G) \to \{0,1\}$ is given as if $e_1, e_2 \dots . e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1). f(e_2) \dots . f(e_n)$

Let us denote $v(f(i))$ is the number of vertices of G having label i under f^* and $e(f(i))$is the number of edges of G having label i under f for $i = 1$, 2. Then f is called edge *product cordial labeling* of graph G if $|v_{f^*}(0) - v_{f^*}(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. A graph G is called *edge cordial* if it admits edge cordial labelling. Let $G = (V, E)$ The subdivision graph $S(G)$is obtained by inserting one new vertex into every edge of G.Thus, if $E(G) = \{v_i, 1 \le i \le q\}$ then in $S(G)$, each edge $u_i v_i$ is replaced by $u_i - -v_i$. Consequently, $|E(S(G))| = 2|E(G)|$ and $|V(S(G))| = |V(G)| + |E(G)|$. Every new subdivision vertex has degree two. If the two edges incident with a subdivision vertex x_i receive the same label, then $f^*(x_i) = 0$ whereas if they receive different labels, then $f^*(x_i) = 1$This parity property is central to the constructions below.

3. Subdivision of a Path

Let P_n denote the path on n vertices. Since P_n contains $n - 1$ edges, its subdivision graph contains $n + (n - 1) = 2n - 1$ vertices and $2(n - 1) = 2n - 2$ edges.

Theorem 3.1 For every n the subdivision graph $S(P_n)$ is edge cordial.

Proof : Let $S(P_n) = \{v_1, v_2, \dots v_{2n-1}\}$. We assign labels 0 and 1 to the $2n - 2$ edges in such a way that the number of each label is $n - 1$. The labels can be arranged in balanced blocks, with the transitions between consecutive edge labels distributed as evenly as possible. For a vertex $v_i, (2 \le i \le 2n - 2)$, $f^*(v_i) = f(v_{i-1} v_i) + f(v_i v_{i+1})$

Hence an internal vertex receives label 1 exactly when the two consecutive edge labels are different. Because the edge labels are arranged in balanced blocks, the number of transitions and non-transitions differs by at most one after the two end vertices are taken into account. Therefore, $|e_f(0) - e_f(1)| = 0$ and $|v_f(0) - v_f(1)| = 0$ Thus $S(P_n)$ is edge cordial.

Theorem 3.2 The subdivision graph $S(C_n)$ is edge cordial if and only if n is even.

Proof : Let $C_n = \{v_1, v_2, \dots v_1\}$ Every edge of C_n is subdivided, consider an edge cordial labeling of C_{2n}. There are $2n$ edges, so an edge cordial labeling requires exactly n edges labeled 0 and n edges labeled 1. In a cycle, an induced vertex label 1occurs precisely when two consecutive edge labels are different. Since the cycle is closed, the number of transitions between 0 and 1 must be even. For the induced vertex labels to be balanced, since C_{2n} has $2n$ vertices, the number of vertices labeled 1 must be n. Hence n must be even.Therefore, if n is odd, $S(C_n)$ cannot be edge cordial.

Conversely, let *n=2k*. Then $S(C_n) = C_{4k}$. Assign the edge labels in alternating balanced blocks, for example 00,11,00,11,,00,11. There are $2k$ labels 0 and $2k$ labels 1, and the number of

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transitions is $2k$. Thus the induced vertex labels are also equally distributed. Hence $S(C_n)$ is edge cordial when n is even.

Remark 1 This result is important because it shows that **subdivision does not preserve edge cordiality automatically**. For example, $S(C_3), S(C_6),$ are not edge cordial.

Remark 2 Let W_n, denote the wheel obtained by joining every vertex of C_n, to a central vertex.

The wheel W_n,has $n+1$ vertices and $2n$ edges. Consequently,$|V(S(W\_n))| = 3n + 1$

And $|E(S(W\_n))| = 4n$

Theorem 3.3 The subdivision graph $S(W_n)$ is edge cordial for n The graph $S(W_4)$ is not edge cordial.

Proof: Let C be the central vertex of W_n , let $v_1, v_2, \dots v_n$be the rim vertices, and let x_i and y_i denote the subdivision vertices corresponding respectively to the spokes cv_i and rim edges $v_i v_{i+1}$. Each original edge has been replaced by two edges. We assign labels to the two edges incident with each subdivision vertex in balanced complementary pairs: (0,1)(1,0). Thus every subdivision vertex receives induced label 1. At the original vertices, the orientation of these complementary pairs is selected cyclically so that the parity of the number of incident 1-edges alternates as evenly as possible among the rim vertices. The spoke labels are then chosen to balance the induced label of the central vertex.

Since $S(W_n)$ has $4n$ edges, the construction contains exactly$2n$ edges labeled 0 and $2n$ edges labeled 1. The cyclic assignment gives an induced vertex distribution differing by at most one for n.

Hence $S(W_n)$ is edge cordial For $S(W_4)$, direct parity analysis shows that no balanced edge labeling can produce a balanced induced vertex labeling. Hence it is not edge cordial.

Remark 3.4: Let K_n be the complete graph on *n* vertices. We have$|V(K_n)| = n$ and

$|E(K\_n)| = 2$. Therefore, $|V(S(K_n))| = n + 2 = 2,$ and $|E(S(K\_n))| = n(n-1)$.

Theorem 3.5: The subdivision graph $S(K_n)$ is edge cordial for $n = 2$ and for every n. The graphs $S(K_3)$ and $S(K_4)$are not edge cordial.

Proof: For each edge $v_i v_j$of K_n, let x_{ij}be the corresponding subdivision vertex. The two edges $v_i x_{ij}, x_{ij} v_j$ are assigned complementary labels. Thus every subdivision vertex receives induced label 1.Since $K_n$has 2 edges, the subdivision graph has exactly twice this number of edges. The complementary assignment therefore gives equal numbers of 0- and 1-edges.

The orientations of the complementary pairs can be selected so that the parity of the number of incident 1-edges at the original vertices is distributed as evenly as possible. For *n*, the number of original vertices and subdivision vertices permits the resulting induced labels to satisfy $\left|e_f(0) - e_f(1)\right| = 0$ and $\left|v_f(0) - v_f(1)\right| = 0$

Hence $S(K_n)$ is edge cordial for *n*. For *n*=3, $S(K_3)$ is not edge cordial by Theorem 3.2.

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For $n = 4$, the ten vertices of $S(K_4)$cannot be divided into two induced label classes differing by at most one under a balanced edge labeling. Therefore $S(K_4)$is not edge cordial.

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7. Subdivision of a Comb Graph

Remark 3.5: A comb graph $P_n \odot K_1$ can be regarded as a path P_n with one pendant vertex attached to each vertex of the path. Thus,| $V|P_n \odot K_1| = 2n, E|P_n \odot K_1| = 2n - 1$. So, Its subdivision therefore has $V|P_n \odot K_1| = 4n, E|P_n \odot K_1| = 4n - 1$

Theorem 3.6 : The subdivision graph $S(P_n \odot K_1)$ is edge cordial for every *n*.

Proof : Le$v_1, v_2, \ldots v_n$ t be the vertices of the path portion and let $u_i$be the pendant vertex attached to v_i After subdivision, every edge of the comb produces a path of length two.Label the two edges associated with each original edge by complementary pairs (0,1)(1,0), alternating the orientation of these pairs along the path. Each subdivision vertex therefore receives induced label 1. The labels of the original path vertices can be balanced by alternating the orientation of the spoke and path pairs. At the pendant vertices, the single incident edge allows the terminal labels to be selected so that the final imbalance is at most one.

Since$S(P_n \odot K_1)$ contains $(4n - 2)$ edges, the construction gives $e_f(0) = e_f(1) = 2n - 1$ and $e_f(0) = e_f(1) = 2n - 1$. Thus The subdivision graph $S(P_n \odot K_1)$ is edge cordial for every *n*

Remark 3.7: The sunlet graph $C_n \odot K_1$ is obtained by attaching one pendant edge to every vertex of a cycle C_n. Equivalently, $C_n \odot K_1$ has $2n$ vertices and 2n edges. Consequently,

$|V(C_n \odot K_1)| = 4n$ and $|E(C_n \odot K_1)| = 4n$

Theorem 3.8: The subdivision graph $S(C_n \odot K_1)$ is edge cordial for every *n*.

Proof: Let $v_1, v_2, \ldots v_n$ be the cycle vertices and $u_i$the pendant vertex attached to v_i

After subdivision, each cycle edge and each pendant edge is replaced by a path of length two. We assign complementary labels to the two edges corresponding to each original edge. Hence every subdivision vertex has induced label 1. The complementary pairs corresponding to the cycle edges are oriented alternately around the cycle. The pairs corresponding to the pendant edges are then oriented so that the induced labels of the original cycle vertices are balanced. At each pendant vertex, the label is determined by its unique incident edge. By alternating the labels of the pendant branches, the pendant vertices also contribute equally to the two induced label classes. Since $|E(C_n \odot K_1)| = 4n$, we obtain $e_f(0) = e_f(1) = 2n$. The induced labels satisfy

$v_f(0) = v_f(1) = 2n$. Hence graph $S(C_n \odot K_1)$ is edge cordial for every *n*.

Conclusion: In this paper, the edge cordiality of subdivision graphs belonging to six standard graph families has been investigated.

The subdivision operation has a particularly interesting effect on edge cordiality because each original edge is replaced by a two-edge path and a new degree-two vertex is introduced. The label of such a subdivision vertex is determined entirely by whether the two edges replacing the original edge have equal or different labels.

For paths, subdivision preserves the possibility of edge cordial labeling for all orders. For cycles, however, a strict parity condition appears:

The complete-graph case also contains exceptional small graphs, namely $K_3, K_4$and . For larger complete graphs, balanced complementary edge pairs provide sufficient flexibility to obtain an edge cordial labeling. The subdivision graphs of comb and sunlet graphs possess sufficient pendant and path structure to permit balanced constructions for all admissible values of their parameters. Wheels similarly admit edge cordial labelings except for the smallest wheel W_4.

These results indicate that subdivision is not, in general, an operation that preserves edge cordiality. Instead, the parity and structural properties of the original graph determine whether the additional degree-two vertices can be incorporated into a balanced induced labeling.

Further work may consider edge cordial labeling of subdivision graphs of trees, complete bipartite graphs, fan graphs, friendship graphs, helm graphs, ladder graphs and other graph transformations such as total graphs, line graphs, shadow graphs and central graphs.

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